REAL OPTION VALUE

CHAPTER 5 MULTI-FACTOR AMERICAN PERPETUAL REAL OPTIONS

In Chapter 4, there is a single underlying state variable, V, which is usually the present value of project future cash flows. In this chapter, either the components of V (or the annualized operating cash flows), or both V and K can be stochastic and divisible. Considering the components of V such as net unit profits and volume is appropriate if the number of units sold and the net profit per unit are affected by different factors. As an example, when demand is inelastic a change in price (and consequently in revenue assuming all other factors are constant) will change the quantity sold but by a smaller proportionate change than when demand is elastic. Correlation between volume and unit profits may vary according to market structure. There may be positive correlation between P and Q, if there is both excess demand and economies of scale.

5.1. STOCHASTIC VALUE AND INVESTMENT COST

In the previous section, it is assumed, implicitly, that investment costs (and possibly variable costs) are fixed. Now, assume that both the net value and investment costs are stochastic and possibly correlated, as in McDonald and Siegel (1986) (M&S), Sick (1989), Williams (1991), Quigg (1993), and Mauer and Ott (1999).

M & S were probably the first to provide a two-stochastic-factor real option investment model by reducing two dimensions to one. Assume that both the net value and investment costs follow a geometric Brownian motion:

$$dV = \alpha_V V dt + \sigma_V V dz_1 \qquad (1)$$

$$dK = \alpha_K K dt + \sigma_K K dz_2 \quad (2)$$

where α is the expected growth or drift of the value and cost, σ the volatility of the value and investment cost, and the Wiener processes dz_1 and dz_2 can be correlated denoted by a parameter $\rho_{V,K}$. Viewing the V and K drift in a risk-neutral world implies that $r-\alpha_V=\delta_V$, the asset or convenience yield, similarly for δ_K.

Although the asset values and the investment cost follow the same type of stochastic process, implying that both variables have associated some sort of uncertainty, the uncertainty related to the investment cost will disappear as soon as the investment is sunk. Let F (V, K), the real option value (ROV), represent the value of the opportunity to invest in a project with stochastic values and investment cost. The partial differential equation (PDE) that explains the movements in the value of the option to invest in the project is:

$$\frac{1}{2}\left[\sigma_V^2 V^2 \frac{\partial^2 F}{\partial V^2} + \sigma_K^2 K^2 \frac{\partial^2 F}{\partial K^2} + 2\rho_{V,K}\sigma_V\sigma_K VK \frac{\partial^2 F}{\partial V \partial K}\right] + (r-\delta_V)V\frac{\partial F}{\partial V} + (r-\delta_K)K\frac{\partial F}{\partial K} - rF = 0 \quad (3)$$

Equation (3) has to be subject to the value matching and the smooth pasting conditions, where $\hat{V}$ and $\hat{K}$ denote the values that justify immediate investment:

$$F(\hat{V},\hat{K}) = \hat{V} - \hat{K} \quad (4)$$

and $\frac{\partial F}{\partial \hat{V}} = 1$ and $\frac{\partial F}{\partial \hat{K}} = -1$ (5)

Using the transformation[1] variable, Z=V/K the option to invest can be written as F(V, K) =K W(Z) where W(Z) is determined from the ordinary differential equation (ODE), where $\eta^2 = \sigma_V^2 + \sigma_K^2 - 2\rho\sigma_V\sigma_K$:

$$\frac{1}{2}(\eta^2)Z^2\frac{\partial^2 W}{\partial Z^2} + (\delta_K - \delta_V)Z\frac{\partial W}{\partial Z} - \delta_K W(Z) = 0 \quad (6)$$

Equation (6) should be subject to the transformed boundary conditions:

$$W(0) = 0 \quad (7)$$

[1] Assuming V and K are homogeneous of degree one, so F(tV, tK)=t F(V,K).

$$W(\hat{Z}) = \frac{\hat{V}}{\hat{K}} - 1 \tag{8}$$

$$\frac{\partial W}{\partial \hat{Z}} = 1 \tag{9}$$

where $\hat{Z}$ is the threshold. Subjecting equation (6) to the appropriate boundary conditions we obtain:

$$\hat{Z} = \frac{\beta_1}{\beta_1 - 1} \tag{10}$$

where β_1 is given by:

$$\beta_1 = \frac{1}{2} - \frac{\delta_K - \delta_V}{\eta^2} + \sqrt{\left(\frac{\delta_K - \delta_V}{\eta^2} - \frac{1}{2}\right)^2 + \frac{2\delta_K}{\eta^2}} \tag{11}$$

$$W(V,K) = AKZ^{\beta_1} \tag{12}$$

$$A = \frac{\hat{Z} - 1}{\hat{Z}^{\beta_1}} \tag{13}$$

$$W(Z) = K(\hat{Z} - 1)(\frac{Z}{\hat{Z}})^{\beta_1} \tag{14}$$

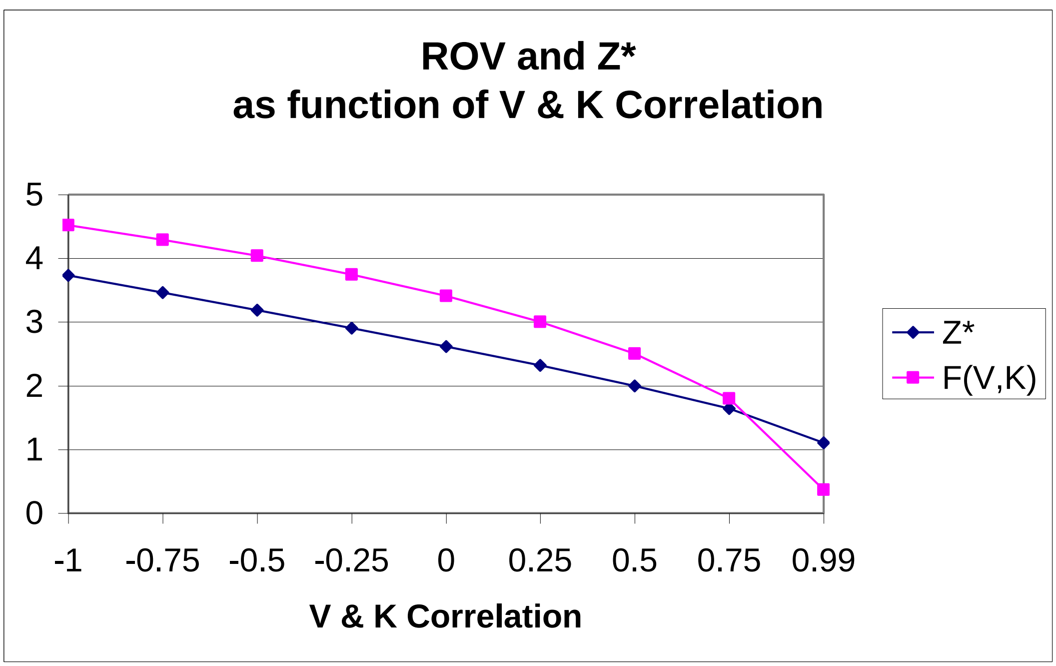
Equation (6) is a second order linear ordinary differential equation with constant coefficients in the form of: $ay'' + by' + cy = 0$ and (b^2-4ac)>0.

With the given yields, volatilities and correlation, the Z (or exchange ratio) volatility is 20% and β_1=2.00. The optimal investment trigger is the ratio of V to K, so that if K remains at 10, $\hat{V}$ =20, but if V remains at 10, $\hat{K}$ =5. Investment is justified if V increases or K decreases, so that $Z > \hat{Z}$. With the given parameter values, the option value is entirely due to uncertainty since V=K, there is nil intrinsic option value.

Figure 1

	A	B	C	D
1	**American Perpetual Exchange Option**			EQ
2	INPUT	Stochastic V and K		
3	V	100		
4	K	100		
5	σ_V	0.20		
6	σ_K	0.20		
7	δ_V	0.04		
8	δ_K	0.04		
9	ρ	0.50		
10	OUTPUT			
11	F(V,K)=W(Z)	25.00	IF(B17<B16,B4*B13*(B17^B14),B12)	12
12	V-K	0.00	B3-B4	
13	A	0.25	(B16-1)/(B16^B14)	13
14	β_1	2.00	0.5-(B8-B7)/(B15^2)+SQRT(((B8-B7)/(B15^2)-0.5)^2 + (2*B8)/(B15^2))	11
15	η	0.20	SQRT(B5^2+B6^2-2*B9*B5*B6)	
16	Z*	2.00	(B14/(B14-1))	10
17	Z	1.00	B3/B4	
18	ODE	0.00	0.5*(B15^2)*(B17^2)*B20+(B8-B7)*B17*B19-B8*B11	6
19	W'(Z)	50.00	B4*B13*B14*(B17^(B14-1))	15
20	W''(Z)	50.00	B4*B13*B14*(B14-1)*(B17^(B14-2))	16
21	W(Z*)	100.00	B13*B4*((B16^B14))	
22	W(Z*)	100.00	(B16-1)*B4	8
23	W'(Z*)	1.00		9
24	W(Z)	25.00	B4*(B16-1)*((B17/B16)^B14)	14

Figure 2



In Figure 2 we can see that the ROV and trigger functions are decreasing functions of correlation, that is the higher the value and investment cost correlation, the lower is the exchange volatility. With low exchange volatility, there is little real option value, but also less downside risk, so investment is justified at lower ratios of V/K. Where volatility is nearly nil, the net present value rule is justified, that is invest if V>K, or V/K >1. Note that the "gamma" coefficients of (6) are $(A) = .5*\sigma_V^2 V^2, (B) = .5*\sigma_K^2 K^2, (C) = \rho_{V,K}\sigma_V\sigma_K VK$, since $B^2 - 4AC$ is negative, this is an elliptical PDE.

Merton (1973) suggests that for a stochastic stock price S, and a constant or deterministic exercise price E, the PDE governing the option price over time might be reduced to an ODE by defining the variable x=S/E, so that the option value function is homogeneous of degree one in {S, E}.

M & S set the disclosure standard of numerical results for different sets of the parameter values, noting that the threshold $\hat{Z}$ and ROV are positive functions of V drift, V and K volatility and negative functions of correlation, and K drift. M & S make two additional comments: that in the sensitivity analysis, it is assumed that any changes in volatilities do not affect the required rates of return or drifts; and that, of course, strictly speaking, it is not reasonable to suppose that project values (or IBM stock) follow geometric Brownian motion, since there would have to be a chance that "IBM would become indefinitely large relative to the economy as a whole". Incidentally, M & S note that swapping one risky asset for another can also be regarded as an asset replacement problem, and also a model for the optimal scrapping of a project (see Adkins and Paxson, 2017 EJF).

When the option value is homogenous of degree 1 in V and K, with Z=V/K, so that W(Z)=F(V,K)/K, Sick (1989) shows that the explicit derivatives when substituted into the PDE, dividing then by K, results in the ODE, and also the value matching and smooth pasting conditions hold. Figure 1 B18 ODE is solved calculating the ROV Δ delta (15), and Γ gamma (16).

$$W'(Z) = KA\beta_1(Z^{\beta_1 - 1}) \tag{15}$$

$$W''(Z) = KA\beta_1(\beta_1 - 1)(Z^{\beta_1 - 2}) \tag{16}$$

In Figure 1, the value matching conditions B21=B22, and the smooth pasting condition B23 holds.

5.2 REVENUE & INVESTMENT COST UNCERTAINTY

Assume that both the net revenue and investment costs follow a geometric Brownian motion:

$$dR = \alpha_R R dt + \sigma_R R dz_1 \tag{17}$$

where α_R is the expected growth or drift of the revenue, σ_R the volatility of the revenue and investment cost, and the Wiener processes dz_1 and dz_2 can be correlated denoted by a parameter $\rho_{R,K}$. Viewing the R and K drift in a risk-neutral world implies that r-α_R=δ_R , the asset or convenience yield, similarly for δ_K.

Dixit and Pindyck (1994) Chapter 6, Section 5, adapt the McDonald and Siegel (1986) model to consider net revenue and investment cost uncertainty.[2] If z=R/K the PDE can be reduced to an ODE, with the substitutions similar to those provided in Sick (1989).

Using the transformation[3] variable, the option to invest can be written as F(R, K)=K W(z) where W(z) is determined from the ordinary differential equation (ODE), where $\nu^2 = \sigma_R^2 + \sigma_R^2 - 2\rho\sigma_R\sigma_K$:

[2] If there is no form of homogeneity, they point out the solution must solve free-boundary problems for elliptic partial differential equations, requiring numerical methods of some complexity.
[3] Assuming R and K are homogeneous of degree one, so F(tR, tK)=t F(R,K).

$$\frac{1}{2}\left(\nu^2\right)z^2\frac{\partial^2 W}{\partial z^2}+\left(\delta_K-\delta_R\right)z\frac{\partial W}{\partial z}-\delta_K W(z)=0 \tag{18}$$

Subjecting equation (18) to the appropriate boundary conditions we obtain:

$$\frac{\hat{R}}{\hat{K}}=\hat{z}=\frac{\beta_1}{\beta_1-1}\delta_R \tag{19}$$

where β_1 is given by:

$$\beta_1=\frac{1}{2}-\frac{\delta_K-\delta_R}{\nu^2}+\sqrt{\left(\frac{\delta_K-\delta_R}{\nu^2}-\frac{1}{2}\right)^2+\frac{2\delta_K}{\nu^2}} \tag{20}$$

$$W(R,K)=AKRz^{\beta_1} \tag{21}$$

$$A=\frac{(\beta_1-1)^{(\beta_1-1)}K^{-(\beta_1-1)}}{\delta_R(\beta_1^{\beta_1})} \tag{22}$$

$$W(z)=K(\hat{z}-1)(\frac{z}{\hat{z}})^{\beta_1}$$

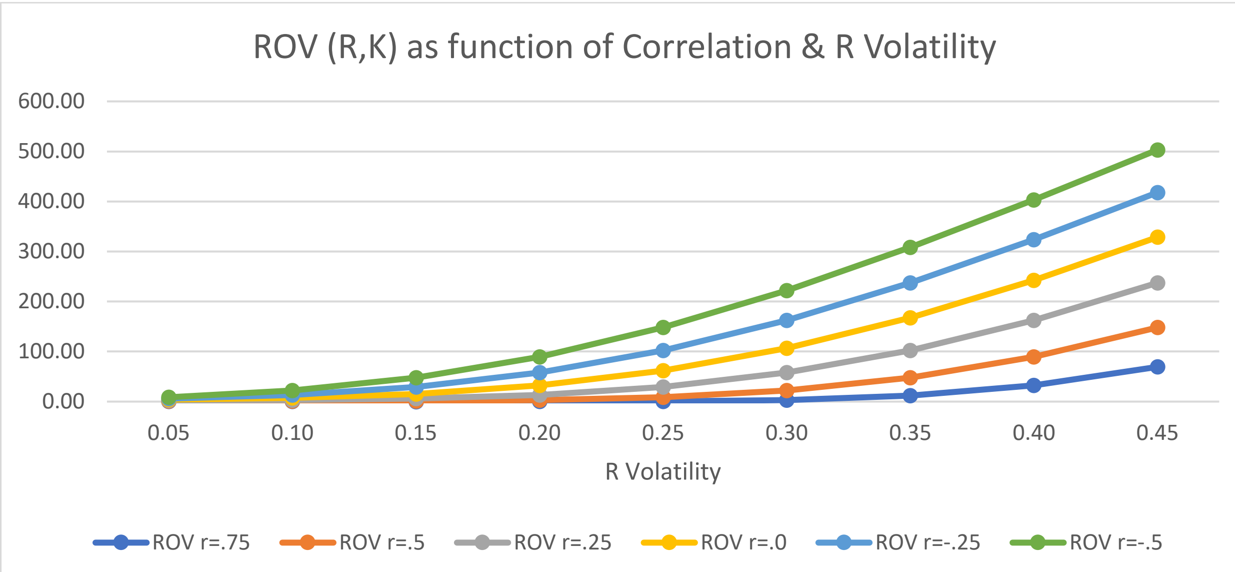
(23)

Figure 3

	A	B	C	D
1	**American Perpetual Exchange Option**			
2	INPUT	Stochastic R and K		
3	R	4.00		
4	K	100		
5	σ_R	0.20		
6	σ_K	0.20		
7	δ_R	0.04		
8	δ_K	0.04		
9	ρ	0.50		
10	OUTPUT			EQ
11	F(R,K)=W(z)	25.00	IF(B17<B16,B13*B4*B3*((B17/B7)^B14),B12)	21
12	R/δ_R-K	0.00	B3/B7-B4	
13	A	0.0625	(((B14-1)^(B14-1))*B4^-(B14-1))/((B7*(B14^B14)))	22
14	β_1	2.00	0.5-(B8-B7)/(B15^2)+SQRT(((B8-B7)/(B15^2)-0.5)^2 + (2*B8)/(B15^2))	20
15	ν	0.20	SQRT(B5^2+B6^2-2*B9*B5*B6)	
16	z*	0.08	(B14/(B14-1))*B7	19
17	z	0.04	B3/B4	
18	ODE	0.00	0.5*(B15^2)*((B17/B7)^2)*B20+(B8-B7)*(B17/B7)*B19-B8*B11	18
19	W'(z)	50.00	B14*B13*B4*B3*((B17/B7)^(B14-1))	
20	W''(z)	50.00	B14*(B14-1)*B13*B4*B3*((B17/B7)^(B14-2))	
21	W(z*)	100.00	B13*B4*B3*((B16/B7)^B14)	
22	W'(z*)	100.00	B14*B13*B4*B3*((B16/B7)^(B14-1))	
23	W(z)	25.00	B13*B4*B3*((B17/B7)^B14)	23

With the given yields, volatilities and correlation, the exchange ratio volatility is 20% and β_1=2.00. With the given parameter values, the option value is entirely due to uncertainty since R/δ_R=K, there is nil intrinsic option value.

Figure 4



Note that while the thresholds and ROV are highly sensitive to R volatility and the correlation of R & K, the vegas are all positive.

5.3 UNIT PROFIT & QUANTITY UNCERTAINTY

Suppose that both the profit per unit and the number of units follow different but possibly correlated geometric Brownian motion processes. Let P represent the profit per unit sold and Q the quantity sold in a market by a firm. Assume that each variable follows a geometric Brownian motion of the form:

$$dP = \mu P\,dt + \sigma P\,dz_1 \quad (24)$$

$$dQ = \omega Q dt + \alpha Q\,dz_2 \quad (25)$$

where μ and ω are the expected multiplicative trends of P and Q, σ and α are the volatilities, and dz_1 and dz_2 the increments of a Wiener process. The two variables may be correlated with correlation coefficient ρ.

Consider a portfolio that consists of a long position in the option to enter a given market, $H(P,Q)$, and a short position consisting of Δ_1 and Δ_2 units of P and Q, respectively. Assume that firms are risk-neutral.[4] Applying Ito's lemma, the following PDE for a firm is obtained (where r=riskfree rate):

$$\frac{1}{2}\frac{\partial^2 H(P,Q)}{\partial P^2}\sigma^2 P^2 + \frac{1}{2}\frac{\partial^2 H(P,Q)}{\partial Q^2}\alpha^2 Q^2 + \frac{\partial^2 H(P,Q)}{\partial P \partial Q}PQ\rho\sigma\alpha$$
$$+ \mu P \frac{\partial H(P,Q)}{\partial P} + \omega Q \frac{\partial H(P,Q)}{\partial Q} - rH(P,Q) = 0 \qquad (26)$$

Equation (26) explains the movements in the value function of a firm with an investment opportunity and is subject to the usual boundary conditions. The first boundary condition is the value matching that gives the value of H(P,Q) at which the firm should invest. The second boundary condition is the smooth pasting that assures that the derivatives of the two functions (before and after the firm enters the market) are equal at the investment point.

Let $X = PQ$ denote the total profit for the firm (no operating cost), implying that $P(X) = H(P,Q)$. After the appropriate substitutions[5], equation (26) can be re-written as:

$$\frac{1}{2}X^2 \frac{d^2 P(X)}{dX^2}\left[\sigma^2 + \alpha^2 + 2\rho\sigma\alpha\right] + X\frac{dP(X)}{dX}\left[\rho\sigma\alpha + \mu + \omega\right] - rP(X) = 0 \qquad (27)$$

Equation (27) is an ODE with the following characteristic quadratic function:

$$\frac{1}{2}(\sigma^2 + \alpha^2 + 2\rho\sigma\alpha)\beta(\beta - 1) + (\rho\sigma\alpha + \mu + \omega)\beta - r = 0 \qquad (28)$$

Equation (28) has two roots, a positive and a negative one, given by:

[4] The assumption of risk neutrality may be relaxed by adjusting the drifts of P and *Q* to account for a risk premium.
[5] See the Appendix for similarity methods.

$$\beta_{1,2} = \frac{1}{z^2}\left(-\left(\rho\sigma\alpha + \mu + \omega - \frac{1}{2}z^2 \right) \pm \sqrt{2rz^2 + \left(\rho\alpha\sigma + \mu + \omega - \frac{1}{2}z^2 \right)^2} \right) \tag{29}$$

where $z^2 = \alpha^2 + \sigma^2 + 2\rho\sigma\alpha$.

The solution of equation (27) is:

$$P(X) = AX^{\beta_1} + BX^{\beta_2} \tag{30}$$

We know that as X increases, the value function of the firm has to increase and that equation (30) has to be finite, thus B equals zero. Equation (30) is subject to the value-matching condition:

$$P(X^*) = \frac{X^*}{r-\mu-\omega} - K \tag{31}$$

where X^* is the firm's trigger value, that is the value of X at which the firm should enter the market, and is also subject to the smooth-pasting condition:

$$\frac{dP(X^*)}{dX} = \frac{1}{r-\mu-\omega} \tag{32}$$

Equations (30), (31) and (32) imply that:

$$X^* = \frac{K(r-\mu-\omega)\beta_1}{\beta_1 - 1} \tag{33}$$

Thus the value function of the firm, H(P,Q), is given by:

$$H(P,Q) = \begin{cases} \frac{K}{\beta_1 - 1}\left(\frac{X}{X^*}\right)^{\beta_1} & X < X^* \\ \frac{X}{r-\mu-\omega} - K & X \geq X^* \end{cases} \tag{34}$$

Equation (34) describes the value function of the firm before and after the trigger is hit. Before the trigger X* is hit, the firm has not yet entered the market and its value function is a monopoly perpetual American option to invest. At the trigger, the firm invests and after that its value function is the net present value.

Figure 5

	A	B	C	D
1	**American Multi-factor Perpetual Real Option**			
2	INPUT	Stochastic P & Q	Paxson & Pinto 2005	
3	P	1.00		
4	Q	2.00		
5	X	2.00		
6	K	100.00		
7	σ	0.20		
8	α	0.20		
9	ρ	-0.50		
10	r	0.04		
11	μ	0.01		
12	ω	0.01		
13	OUTPUT			
14	ν	0.02	B10-B11-B12	
15	H(P,Q)	25.00	IF(B5<B19,(B6/(B17-1))*((B5/B19)^B17),B16)	34
16	X/ν-K	0.00	B5/B14-B6	
17	β_1	2.00		29
18	σ_X	0.20	SQRT(B7^2+B8^2+2*B9*B7*B8)	
19	X*	4.00	B6*B14*(B17/(B17-1))	33
20	ODE	0.00	0.5*(B18^2)*((B5/B14)^2)*B22+(B9*B7*B8+B11+B12)*(B5/B14)*B21-B10*B15	27
21	H'(P,Q)	0.50	B23*B17*((B5/B14)^(B17-1))	35
22	H''(P,Q)	0.01	B23*B17*(B17-1)*(B5/B14)^(B17-2)	
23	A	0.0025	((B19/B14)-B6)/((B19/B14)^B17)	
24	ROV	25.00	B23*(B5/B14)^B17	

Figure 5 parameters are chosen so that the real option results and optimal X* which justifies making the investment are similar to Figure 4.1 (times 100 for V and K). Note that $r-\mu-\omega=\nu$, so $V=X/\nu$ and $V^*=X^*/\nu$. With suitable drifts, volatilities and correlation parameters, the net profit volatility z=20% and $\beta_1=2.00$. So, the optimal V* equivalent is the same as in Figure 4.1, and the ROV=H(P,Q) is the same for the same level of investment cost and V equivalent.

The first derivative (delta) of the value function of the firm, where the number of units and the profit per unit are the state variables, is:

$$\frac{dH(P,Q)}{dX} = \begin{cases} \frac{1}{r-\mu-\omega}\left(\frac{X}{X^*}\right)^{\beta_1-1} > 0 & X < X^* \\ \frac{1}{r-\mu-\omega} > 0 & X \geq X^* \end{cases} \quad (35)$$

Delta behaves as expected, that is as total profit increases the firm's ROV also increases, until it reaches a constant 1/ν at X=X*.

Figure 6

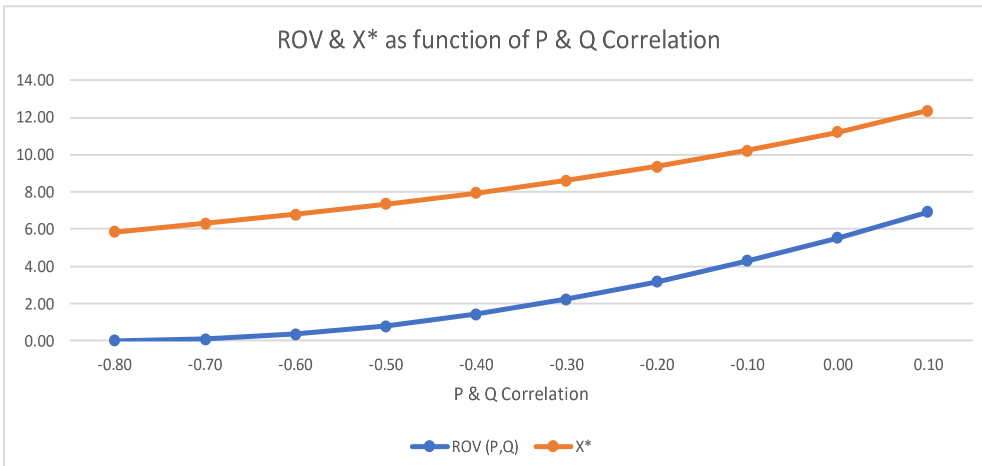


Figure 6 shows the sensitivity of the ROV and the X threshold to changes in the correlations of P and Q.

The primary difference between the American multi-factor perpetuity and the "vanilla" American perpetuity is the separate consideration of P and Q drifts, separate volatilities and correlation between P and Q. Typically, economists model inverse demand curves, so that in normal markets Q= 1- θP. As the price declines Q increases, or with fixed demand, as Q increases P decreases. Thus P and Q are negatively correlated.

Figure 7

ROV (P,Q)	11.71	8.93	7.74	7.76	8.57	9.85	11.39	13.05	14.75	16.44
X*	6.15	5.48	5.22	5.22	5.40	5.70	6.07	6.49	6.96	7.46

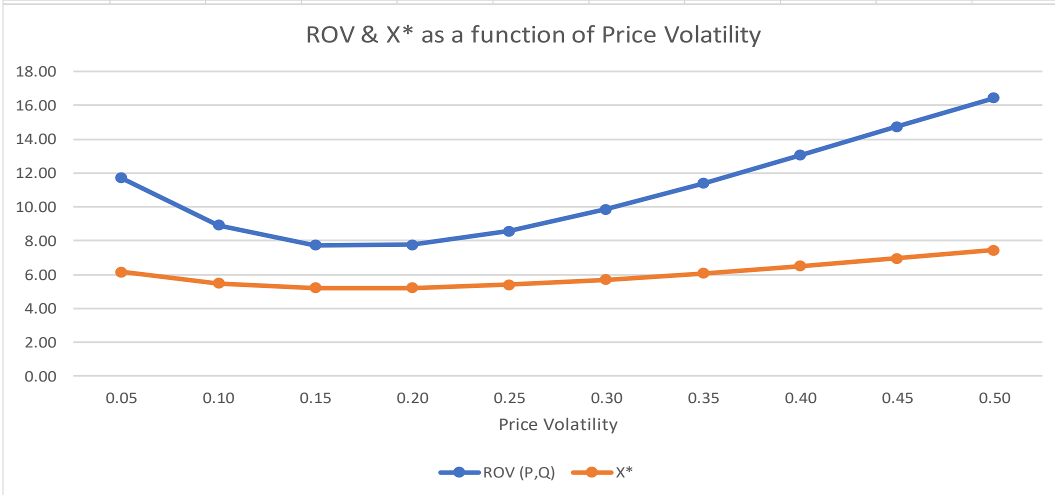


Figure 7 illustrates the sensitivity of both the ROV=H(P,Q) and X* to increases in price volatility, with negative correlation between P and Q, and low or nil P and Q drift. In this case both ROV and X* threshold vegas are negative as price volatility increases from a low level, and then eventually become positive at high price volatility.

SUMMARY

This chapter presents real perpetual American multi-factor option models. A firm maximises the value of its investment decision not when the present value of the cash flows equals the investment cost, but when V/K is much greater than one, unless there is little volatility in V and K, and V and K are highly correlated. The ROV and Z* are derived as the solution to an ordinary differential equation. It is easy, especially in Excel, to show that the solution, along with the first and second derivatives of the ROV, do actually solve the differential equation. Multi-factor models are able to cover estimations of several state variables, the volatilities of those variables, and correlations among the variables, if warranted. Replicating these real options along a time frame might be attempted using a variety of real, financial and commodity securities, or eventually synthetic or virtual products created by imaginative enterprises.

EXERCISES

EXERCISE 5.1 A bungalow in Putney has a restrictive covenant requiring the permission of the adjacent house owner in order to convert the bungalow into a modern house. Provisionally, a house of 3,000 square feet is envisioned (depends on design), which currently would be worth £300 per square foot, and costs £273 per square foot to build, including demolition, design and other costs. The volatility of Putney houses is 20%, interest rates 4% and expected payout 4%; construction costs are expected to have a 20% volatility and 4% payout, and are 50% correlated with Putney house prices. What is the value of this bungalow site? At what house value should the construction start?

EXERCISE 5.2 Genzyme Corporation has a perpetual option to acquire Blockbuster Ltd. which is currently worth $300 million, in exchange for Genzyme Biosurgery, which is currently worth $300 million. The volatility of Blockbuster is 112%, Biosurgery 95%, the correlation between the companies is 0%, and both have a payout of 2%. What is the value of this option, and at what Blockbuster value should it be exercised?

EXERCISE 5.3 ROGroupie Co. has the opportunity to build a Real Options Network, which will have a revenue of Dedicated Subscribers each paying a DSP price. It cost nothing to operate this network, but the investment cost per expected DS is 100. Suppose the expected number of Dedicated Subscribers is 2, the DSP is 1.00, the volatilities of both DS and DSP are 20% and the drifts 1%, the correlation is -50% and the interest rate is 4%. What is the value of this opportunity, and at what DS*DSP amount should those groupies start this venture?

PROBLEMS

PROBLEM 5.4 A bungalow in Putney has a restrictive covenant requiring the permission of the adjacent house owner in order to convert the bungalow into a modern house. Provisionally, a house of 3,000 square feet is envisioned (depends on design), which currently would be worth £300 per square foot, and costs £273 per square foot to build, including demolition, design and other costs. The volatility of Putney houses is 30%, interest rates 5% and expected yield 3%; construction costs are expected to have a 20% volatility and 2% payout, and due to the influx of new EU workers are not correlated with Putney house prices. What is the value of this bungalow site? At what house value should the construction start?

PROBLEM 5.5 Young Mandy Ma is offered a lifetime investment opportunity in a residential electricity solar facility which currently costs $25,000. The quantity of electricity generated annually forever is around 20 units, which can be sold to the grid (or consumed) at a current average price of 50. V ($P*Q/\delta_V$) and K have a -.50 correlation, both are expected to increase 2% per year and both are highly volatile at 20%. What is the value of this opportunity, and at what electricity price should she exercise the option?

PROBLEM 5.6 The State of Arizona wants to encourage more residential solar, and is considering a subsidy to reduce the investment cost in the previous problem. What is the cash subsidy that would justify immediate installation of Mandy's facility?